8.3 Digital PAM with Noise

8.13. In this section, the transmitted signal is still in the form of digital PAM as in the previous subsection:

$$x(t) = \sum_n m[n]p(t - nT_s)$$

However, here, we also consider the effect of additive noise. Therefore, the received signal is

$$y(t) = x(t) + N(t)$$

where $N(t)$ is a random noise process.

p(t) is a Nyquist pulse.

$x(t)\big|_{t=5T} \equiv x[5] = m[5]$

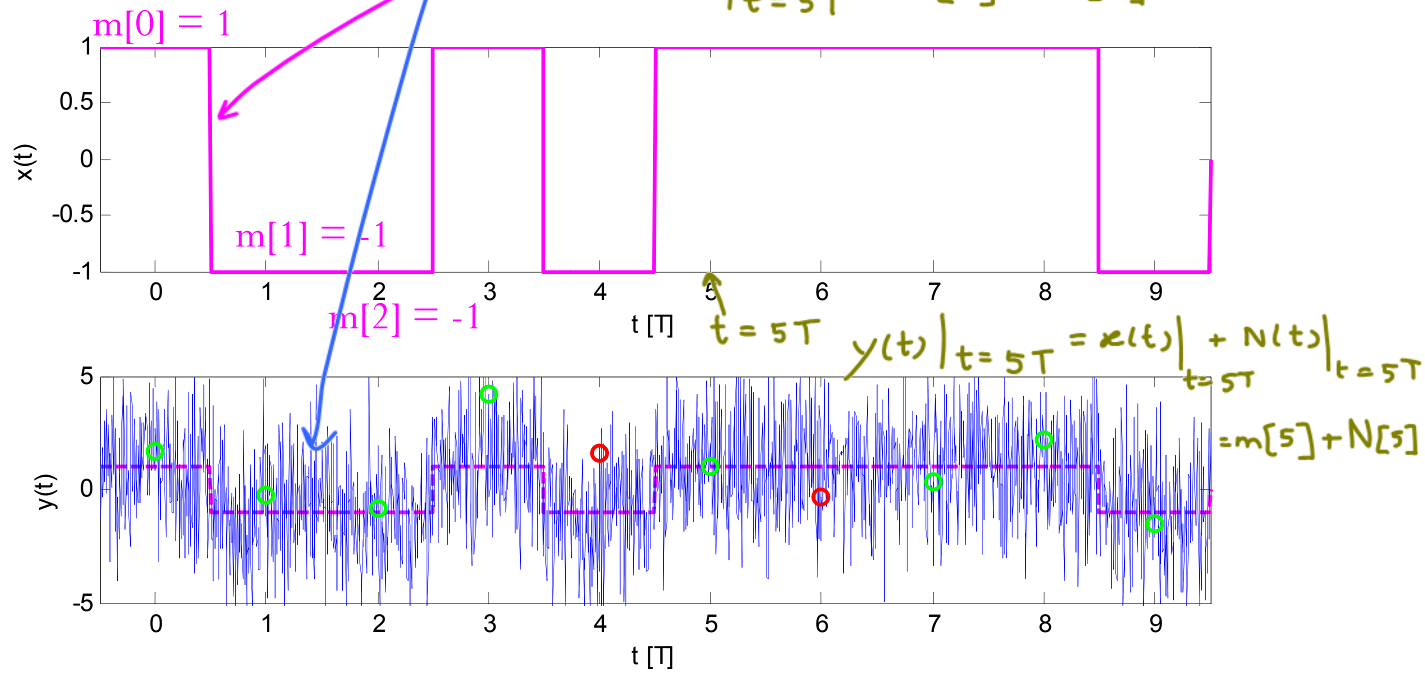


Figure 47: Digital PAM with Noise

8.14. Note that

- The noise $N(t)$ is random.
- The message $m[n]$ should be random (at least from the perspective of the receiver; if the receiver had known in advance the value of $m[n]$, there would have been no point of transmitting $m[n]$).

∘ This makes $x(t)$ and $y(t)$ random.

To emphasize the randomness in the signals under consideration, we sometimes write $M[n]$, $X(t)$, and $Y(t)$ using capital letters[26].

8.15. Simple receiver: For simplicity, let's assume that our receiver simply samples $y(t)$ every T seconds to get

p(t) is a Nyquist pulse

Represented by circles in the Figure

$$y[n] = y(t)|_{t=nT}. = \left(x(t) + N(t)\right)\Big|_{t=nT} = m[n] + N(t)\Big|_{t=nT}$$

m[n]

a

0

−a

threshold

When the alphabet $\mathcal{A}$ contains only two symbols of opposite sign ($\mathcal{A} = \{-a, a\}$, where $a > 0$), the decoded value $\hat{m}[n]$ of our $m[n]$ can be found by

$$\hat{m}[n] = \begin{cases} a, & y[n] \geq 0, \\ -a, & y[n] < 0. \end{cases}$$

y[n] is closer to "a"

y[n] is closer to "−a"

Here we use "0" as the **threshold** level/value. Turn out that this middle point is the *optimal* threshold to use when the two possible symbol values from the alphabet are equally likely.

minimum distance decoding

Example 8.16. In Figure 47, $\mathcal{A} = \{-1, 1\}$.

Idea: noise is usually small.

n	0	1	2	3	4	5	6	7	8	9
$m[n]$	1	-1	-1	1	-1	1	1	1	1	-1
$y[n]$	1.69	-0.27	-0.81	4.20	1.58	1.04	-0.34	0.35	2.19	-1.51
$\hat{m}[n]$	1	-1	-1	1	1	1	-1	1	1	-1

error

Note that the value of the noise $N(t)$ at $t = 4T$ is too positive. Even when "-1" was transmitted, the received value is 1.58 which exceeds 0. Therefore, we get an error at $n = 4$.

Similarly that the value of the noise $N(t)$ at $t = 6T$ is too negative. Even when "1" was transmitted, the received value is -0.34 which is lower than 0. Therefore, we get an error at $n = 6$.

Among the ten symbols sent in this example, there are two symbol errors. Therefore, the **symbol error rate** (SER) or symbol error probability is 2/10.

Because there are two symbols in the alphabet, each symbol transmission conveys 1 bit. Hence, the **bit error rate** (BER) or bit error probability is also 2/10.

[26]Caution: Here, capital letters represent random variables/processes. In earlier sections, we used capital letter to represent Fourier transform. However, we won't talk about Fourier transform here; so confusion can be avoided.

(AWGN)

Normal

8.17. Additive White Gaussian Noise: At each time instant t, the noise $N(t)$ is usually modeled by a Gaussian random variable with mean 0 and standard deviation σ_N,

pdf

$$f_{N(t)}(n) = \frac{1}{\sqrt{2\pi}\sigma_N} e^{-\frac{1}{2}\left(\frac{n}{\sigma_N}\right)^2}.$$

Furthermore, the "white" part means that the noise values at different time instants are independent.

Definition 8.18. In general, a **Gaussian (normal) random variable** X with mean m and standard deviation σ is characterized by its probability density function (PDF):

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}.$$

To talk about such X, we usually write $X \sim \mathcal{N}(m, \sigma^2)$. Probability involving X can be evaluated by

$$P[X \in A] = \int_A f_X(x)dx.$$

In particular,

$$P[X \in [a,b]] = \int_a^b f_X(x)dx = F_X(b) - F_X(a)$$

where $F_X(x) = \int_{-\infty}^x f_X(t)dt$ is called the cumulative distribution function (CDF) of X.

We usually express probability involving Gaussian random variable via the Q function which is defined by

qfunc

$$Q(z) = \int_z^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx.$$

Note that $Q(z)$ is the same as $P[Z > z]$ where $Z \sim \mathcal{N}(0,1)$; that is $Q(z)$ is the probability of the "tail" of $\mathcal{N}(0,1)$.

It can be shown that

- Q is a decreasing function

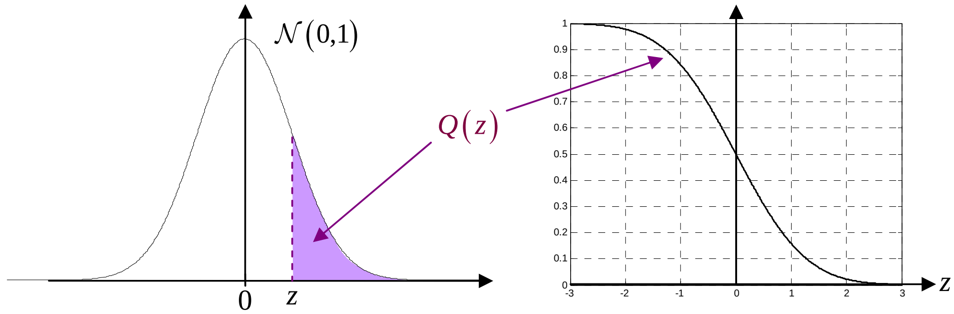
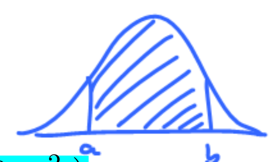


Figure 48: Q-function

- $Q(0) = \frac{1}{2}$
- $Q(-z) = 1 - Q(z)$
  - This is useful for converting the argument of the Q function to positive value.
- For $X \sim \mathcal{N}(m, \sigma^2)$,

$$P[X > c] = Q\left(\frac{c-m}{\sigma}\right).$$



8.19. Three important noise probabilities for $N \sim \mathcal{N}(0, \sigma_N^2)$:

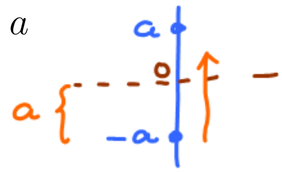
$$P[N > c] = Q\left(\frac{c}{\sigma_N}\right),\ P[N < c] = \overset{1-P[N \geq c]}{1 - Q\left(\frac{c}{\sigma_N}\right)},\ P[a < N < b] = P[N<b] - P[N<a]$$
$$= P[N>a] - P[N>b]$$
$$= Q\left(\frac{a}{\sigma_N}\right) - Q\left(\frac{b}{\sigma_N}\right)$$

Note that all strict inequalities above can also be replaced by the ones that also include equalities because the noise is a continuous random variable and hence including one particular noise value does not change probability.

8.20. For the simple receiver in 8.15, suppose $N(t) \sim \mathcal{N}(0, \sigma^2)$.

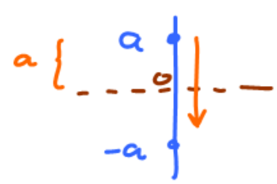
(a) When a $\overbrace{\text{“}-a\text{” was transmitted}}^{\text{event } B^c}$, error occurs when $N(t) > a$

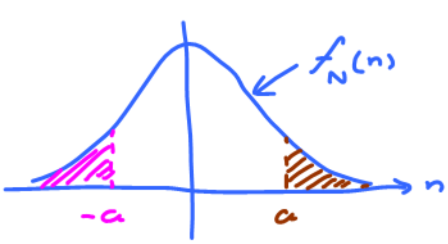
$$P[N(t) > a] = Q\left(\frac{a}{\sigma}\right)$$



(b) When a $\overbrace{\text{“}+a\text{” was transmitted}}^{\text{event } B}$, error occurs when $N(t) < -a$

$$P[N(t) < -a] = 1 - P[N(t) \geq -a]$$
$$= 1 - Q\left(\frac{-a}{\sigma}\right)$$
$$= 1 - \left(1 - Q\left(\frac{a}{\sigma}\right)\right) = Q\left(\frac{a}{\sigma}\right)$$





A formula that connects these events with the (combined) error probability is called the ***Total Probability Theorem***: If a (finite or infinitely) countable collection of events $\{B_1, B_2, \ldots\}$ is a partition of Ω, then

$$P(A) = \sum_i P(A|B_i)P(B_i). \tag{62}$$

(decoding) error event $\mathcal{E}$

In particular,

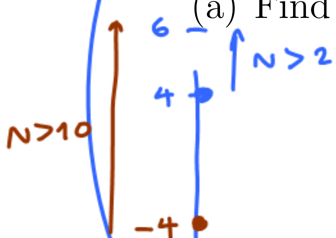
$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c).$$

Here, we replace event A by the error event $\mathcal{E}$. Event B is defined to be the event that the transmitted symbol is "a". The error probability is then

$$P(\mathcal{E}) = \underbrace{P(\mathcal{E}|B)}_{Q\left(\frac{a}{\sigma}\right)}P(B) + \underbrace{P(\mathcal{E}|B^c)}_{Q\left(\frac{a}{\sigma}\right)}P(B^c) = Q\left(\frac{a}{\sigma}\right)\underbrace{\left(P(B) + P(B^c)\right)}_{1}$$

Example 8.21. In a digital PAM system, equally-likely symbols are selected from an alphabet set $\mathcal{A} = \{\overset{-a}{-4}, \overset{a}{4}\}$. The pulse used in the transmitted signal is a Nyquist pulse. The additive noise at each particular time instant is Gaussian with mean 0 and standard deviation 2.

(a) Find the probability that the received signal at a particular time is > 6.



"a" was transmitted; event A

$$P(A|B) = P[N > 2] = Q\left(\frac{2}{2}\right) = Q(1) \approx 0.1587$$

$$P(A|B^c) = P[N > 10] = Q\left(\frac{10}{2}\right) = Q(5) \approx 2.9 \times 10^{-7}$$

$$P(A) = P(A|B)\overset{0.5}{P(B)} + P(A|B^c)\overset{0.5}{P(B^c)} = \frac{Q(1) + Q(5)}{2} \approx 0.0793$$

(b) Find the symbol error probability when 0 is used as the threshold level for the decoding decision at the receiver.

$$P(\mathcal{E}) = Q\left(\frac{a}{\sigma}\right) = Q\left(\frac{4}{2}\right) = Q(2) \approx 0.0228$$

SNR = signal-to-noise (power) ratio $= \dfrac{a^2}{\mathbb{E}[N^2(t)]} = \dfrac{a^2}{\sigma^2}$

$$P(\mathcal{E}) = Q\left(\sqrt{SNR}\right)$$

$$\operatorname{Var} X = \mathbb{E}[X^2] - (\mathbb{E}X)^2$$

$$\mathbb{E}[X^2] = \operatorname{Var} X + (\mathbb{E}X)^2 = \sigma_X^2 + (\mathbb{E}X)^2$$